ELSEVIER

Discrete Mathematics 165/166 (1997) 11-20

DISCRETE MATHEMATICS

D_λ-cycles in λ-claw-free graphs[1]

Mohamed El Kadi Abderrezzak *, Evelyne Flandrin, Amel Harkat-Benhamdine, Hao Li

L.R.I., URA 410 C.N.R.S. Bât. 490, Université de Paris-sud 91405-Orsay cedex, France

Abstract

For a given integer $\lambda \geqslant 1$, we define a λ-claw as a graph H which is connected and contains a vertex w such that $H-w$ is the union of three pairwise remote connected subgraphs with exactly λ vertices each. We define a graph to be *λ-claw-free* if it contains no induced subgraph isomorphic to a λ-claw. We show that if G is a k-connected (k integer, $k \geqslant 2$), λ-claw-free graph such that if for every two remote connected subgraphs S_1, S_2 of G of order λ we have $d(S_1)+d(S_2) > n-(k+1)\lambda-k$, or if for every three pairwise remote, connected subgraphs S_1, S_2, S_3 of G of order λ we have $d(S_1)+d(S_2)+d(S_3) > n-(k+1)\lambda+1$, then G contains a cycle C such that every component of $G-V(C)$ has at most $\lambda-1$ vertices.

1. Introduction and notation

All the graphs considered in this paper are undirected and simple.

Let G be a graph and H a subgraph of G. $G[H]$ represents the subgraph of G induced by H. The neighborhood in H of a vertex u is denoted by $N_H(u)$ and the degree of u in H by $d_H(u)$. If X is a subset of $V(G)$, let $N_H(X) = \cup_{v \in X} N_H(v)$ and $d_H(X) = |N_H(X) - X|$. In the case when $H = G$, we use $N(u)$, $d(u)$, $N(X)$ and $d(X)$ instead of $N_G(u)$, $d_G(u)$, $N_G(X)$ and $d_G(X)$, respectively. If Y is a subset of $V(G)-X$, let $E(X,Y)$ be the set of edges between X and Y and $e(X,Y) = |E(X,Y)|$. We will say that X and Y are remote if $X \cap Y = \emptyset$ and $e(X,Y) = 0$.

In [6], Veldman defined a cycle C in G to be a *D_λ-cycle* (λ integer, $\lambda \geqslant 1$) if every components of $V(G)-C$ has order less than λ and graphs containing a D_λ-cycle to be *D_λ-cyclic*. He gave some sufficient conditions for a graph to be D_λ-cyclic generalizing Chvátal–Erdős' theorem [2] and Dirac's theorem [3] which are obtained as corollaries when $\lambda = 1$. He also conjectured the following which was proved by Fraisse in [5].

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[1] The work was partially supported by PRC MathInfo.

* Corresponding author. E-mail: elkadi@lri.fr.

PII S0012-365X(96)00157-4

Theorem 1. *Let k and λ be positive integers with $k \geqslant 2$ and $\lambda \geqslant 1$. Let G be a k-connected graph of order n such that, for every $k+1$ pairwise remote connected subgraphs $H_0, H_1, \dots, H_k$ of order λ of G, one of the two following propositions is verified:*

$$\sum_{i=0}^{k} d(H_i) > n - (k+1)\lambda + k^2 \quad \text{if } \lambda \geqslant k,$$

$$\sum_{i=0}^{k} d(H_i) > \frac{k+1}{\lambda+1}(n-\lambda) \quad \text{if } \lambda \leqslant k,$$

then G is D_λ-cyclic.

In a different direction many results have been obtained this recent years concerning hamiltonian cycles in claw-free graphs, that is graphs without induced subgraphs isomorphic to $K_{1,3}$. Most of those results appear in the survey [4]. Let us mention the following of Zhang [7].

Theorem 2. *Let G be a k-connected ($k \geqslant 2$) claw-free graph of order n such that the degree sum of every $k+1$ independent vertices is at least $n-k$. Then G is hamiltonian.*

The value $n-k$ is lower than for general graphs (see for example the nonhamiltonian bipartite complete graph $K_{p,p+1}$, $p \geqslant 2$)

In this paper, we generalize claws as follows. Given the integer $\lambda \geqslant 1$, the graph H is said to be a λ-claw if H is connected and contains a vertex w such that $H - w$ is the union of three pairwise remote connected subgraphs with exactly λ vertices each. We define a graph to be *λ-claw-free* if it contains no induced subgraph isomorphic to a λ-claw. It is interesting to know how the results concerning D_λ-cycles can be improved if we also assume that the graph is λ-claw-free. We get the following result.

Theorem 3. *Let k and λ be positive integers with $k \geqslant 2$ and $\lambda \geqslant 1$. Let G be a k-connected, λ-claw-free graph. If for every two remote connected subgraphs S_1, S_2 of G of order λ we have $d(S_1)+d(S_2) > n-(k+1)\lambda-k$, or if for every three pairwise remote, connected subgraphs S_1, S_2, S_3 of G of order λ we have $d(S_1)+d(S_2)+d(S_3) > n-(k+1)\lambda+1$, then G is D_λ-cyclic.*

Notice that if $k=2$ and if we consider three pairwise remote connected subgraphs S_1, S_2, S_3 of G of order λ, we get a better bound than in Theorem 1. But in the special case when $\lambda = 1$, Theorem 3 gives a bound which is one more than in Theorem 2.

2. Proof of Theorem 3

Suppose that G satisfies the hypothesis of Theorem 3 and has no D_λ-cycle.

If $C = c_1c_2\cdots c_pc_1$ is a cycle, we let $C[c_i,c_j]$, be the subpath $c_ic_{i+1}\cdots c_j$, and $\overline{C}[c_j,c_i] = c_jc_{j-1}\cdots c_i$, where the indices are taken modulo p. The cycle C has an implicit orientation according the increasing subscripts and for any i, $1\leqslant i\leqslant p$, we put $c_i^+ = c_{i+1}$, $c_i^- = c_{i-1}$. Similarly, for a subset X of $V(C)$, we define $X^- = \{x^-|x\in X\}$ and $X^+ = \{x^+|x\in X\}$.

If there is no ambiguity, we will often identify subgraphs with their vertex set.

Let $C := c_1c_2\cdots c_pc_1$ be a cycle in G such that:

(a) the cardinality of the largest component in $R := G - C$ is as small as possible;

(b) the number of components in R with the largest cardinality is as small as possible.

For any two vertices x and y in G, denote by xP_Ry a path connecting x and y that is internally disjoint from C. For any vertex $c_i\in C$, put $R(c_i)$=\{ $w\in R$ / there exists a path $c_iP_Rw\} \cup \{c_i\}$.

Let H be a largest component in R with $|H|\geqslant\lambda$. Instead of the above notation xP_Ry, we will use xHy when all the internal vertices of the path connecting x and y are in H. Let $N(H) - H := \{x_1,x_2,x_3,\ldots,x_d\}\subset V(C)$. Clearly $d\geqslant k$. For any vertex $c_i\in C$, considering the indices modulo d, we let

$$f^+(i) = \min\{j\geqslant i/|G[R(c_i)\cup R(c_{i+1})\cup\cdots\cup R(c_j)]|\geqslant\lambda\}.$$

$$f^-(i) = \max\{j\leqslant i/|G[R(c_i)\cup R(c_{i-1})\cup\cdots\cup R(c_j)]|\geqslant\lambda\}.$$

$$D_\lambda^+(c_i) = G[R(c_i)\cup R(c_{i+1})\cup\cdots\cup R'(c_j)],$$

where $j = f^+(i)$ and $G[R'(c_j)]$ is a connected subgraph of $G[R(c_j)]$ such that $c_j\in R'(c_j)$ and $|D_\lambda^+(c_i)| = \lambda$. Similarly we define $D_\lambda^-(c_i)$.

Also for every j, $1\leqslant j\leqslant d$, we choose a connected subgraph of H of cardinality λ that contains a neighbor of x_j and call it $H_\lambda(x_j)$.

We now give some preliminary results:

Claim 1. (a) *$H_\lambda(x_i)$ and $D_\lambda^+(x_i^+)$ are remote.*

(b) *$H_\lambda(x_i)$ and $D_\lambda^-(x_i^-)$ are remote.*

(c) *$D_\lambda^+(x_i^+)$ and $D_\lambda^+(x_j^+)$ are remote.*

Proof. Claim 1 follows immediately from Lemma 1 in [5] and from the proof of Theorem 2 in [6]. □

Remark 1. By Claim 1, λ-claw-free property and by considering $G[x_i\cup H_\lambda(x_i)\cup D_\lambda^+(x_i^+)\cup D_\lambda^-(x_i^-)]$, we deduce that $D_\lambda^+(x_i^+)$ and $D_\lambda^-(x_i^-)$ are not remote.

Let $x_i^{(-)}$ be the last vertex on C before x_i such that there exists a path $x_i^{(-)}P_Ry$ for some $y\in D_\lambda^+(x_i^+)$ and let $x_i^{(+)}$ be the first vertex on C after x_i such that there exists a path $x_i^{(-)}P_Rx_i^{(+)}$.

Remark 2. According to the definition of $x_i^{(-)}$ and $x_i^{(+)}$ and Remark 1, we deduce that

$$|R(x_i^{(-)+}) \cup R(x_i^{(-)++}) \cdots \cup R(x_i^-)| < \lambda,$$
$$|R(x_i^+) \cup R(x_i^{++}) \cdots \cup R(x_i^{(+)-})| < \lambda.$$

Remark 3. According to Remark 2, $x_i^{(-)} \in D_\lambda^-(x_i^-)$ and $x_i^{(+)} \in D_\lambda^+(x_i^+)$. Then, using Claim 1(c), we get

$$x_{i+1}^{(-)} \in C[x_i^{(+)+}, x_{i+1}^-].$$

Claim 2. (a) *For each $i \in \{1,2,3,\ldots,d\}$, for any $u \in C[x_i^{(-)+}, x_i^-]$ and any $v \in C[x_i^+, x_i^{(+)-}]$, there is no path uP_Rv.*

(b) *For each $i \neq j$, for any $u \in C[x_i^{(-)}, x_i^{(+)-}] - \{x_i\}$ and any $v \in C[x_j^{(-)}, x_j^{(+)-}]$, there is no path uP_Rv.*

Proof. (a) Follows directly from the definitions of $x_i^{(-)}$ and $x_i^{(+)}$.

To prove (b), let $i \neq j$ and assume that there exists a path $u_iP_Rv_j$ for some $u_i \in C[x_i^{(-)}, x_i^{(+)-}] - \{x_i\}$ and some $v_j \in C[x_j^{(-)}, x_j^{(+)-}]$. One of the two main cases must occur.

Case $u_i \in C[x_i^{(-)}, x_i^-]$: Without loss of generality we assume that u_i is the last vertex before x_i which is connected to some $v_j \in C[x_j^{(-)}, x_j^{(+)-}]$ by a path $u_iP_Rv_j$.

If $v_j \in C[x_j^{(-)}, x_j^-]$, let

$$C' := x_jHx_iC[x_i, v_j]v_jP_Ru_i\overline{C}[u_i, x_j].$$

According to the choice of u_i and v_j, the subgraphs $G[R(u_i^+) \cup R(u_i^{++}) \cup \cdots \cup R(x_i^-)]$ and $G[R(v_j^+) \cup R(v_j^{++}) \cup \cdots \cup R(x_j^-)]$ are remote. From Remark 2, we deduce that $|R(u_i^+) \cup R(u_i^{++}) \cup \cdots \cup R(x_i^-)| < \lambda$ and $|R(v_j^+) \cup R(v_j^{++}) \cup \cdots \cup R(x_j^-)| < \lambda$. Then C' contradicts the choice of C.

If $v_j \in C[x_j, x_j^{(+)-}]$, let

$$C'' := x_jHx_iC[x_i, x_j^{(-)}]x_j^{(-)}P_Rx_j^{(+)}C[x_j^{(+)}, u_i]u_iP_Rv_j\overline{C}[v_j, x_j].$$

C'' gives a similar contradiction.

Case $u_i \in C[x_i^+, x_i^{(+)-}]$: Without loss of generality we assume that u_i is the first vertex after x_i which is connected to some $v_j \in C[x_j^{(-)}, x_j^{(+)-}]$ by a path $u_iP_Rv_j$. If $v_j \in C[x_j, x_j^{(+)-}]$, the proof is similar as in the case $u_i \in C[x_i^{(-)}, x_i^-]$ and $v_j \in C[x_j^{(-)}, x_j^-]$. If $v_j \in C[x_j^{(-)}, x_j^-]$, the proof is similar as in the case $u_i \in C[x_i^{(-)}, x_i^-]$ and $v_j \in C[x_j, x_j^{(+)-}]$. □

Claim 3. *For each $i \in \{1,2,3,\ldots,d\}$, $C[x_i^{(+)+}, x_{i+1}^{(-)-}] \neq \emptyset$.*

Proof. If the Claim is not true, let

$$C' := x_{i+1}Hx_iC[x_i, x_i^{(+)}]x_i^{(+)}P_Rx_i^{(-)}\overline{C}[x_i^{(-)}, x_{i+1}^{(+)}]x_{i+1}^{(+)}P_Rx_{i+1}^{(-)}C[x_{i+1}^{(-)}, x_{i+1}].$$

From Claim 2(b), the subgraphs $G[R(x_i^{(-)+}) \cup R(x_i^{(-)++}) \cup \cdots \cup R(x_i^-)]$ and $G[R(x_{i+1}^+) \cup R(x_{i+1}^{++}) \cup \cdots \cup R(x_{i+1}^{(+)-})]$ are remote and according to Remark 2, they have less than λ vertices in each. Then the cycle C' contradicts the choice of C. $\square$

Claim 4. *For each $i \in \{1,2,3,\ldots,d\}$, there exists $y_i \in C[x_i^{(+)+}, x_{i+1}^{(-)-}]$ such that $x_{i+1}^{(-)} \notin D_\lambda^+(y_i)$ and for any $u \in D_\lambda^+(y_i) \cap C$ and any $v \in C[x_i^{(-)}, x_i^{(+)-}]$, there is no path uP_Rv.*

Proof. We know from Claim 3 that $C[x_i^{(+)+}, x_{i+1}^{(-)-}] \neq \emptyset$. Let y be the last vertex in $C[x_i^{(+)}, x_{i+1}^{(-)-}]$ which is connected to some $d_i \in C[x_i^{(-)}, x_i^{(+)-}]$ by a path yP_Rd_i. Let w be the first vertex in $C[y^+, x_{i+1}^{(-)}]$ which is connected to some vertex $u \in C[x_{i+1}^{(-)+}, x_{i+1}^{(+)-}]$ by a path wP_Ru. One of the four cases must occur.

If $u \in C[x_{i+1}^{(-)+}, x_{i+1}]$ and $d_i \in C[x_i, x_i^{(+)-}]$, let

$$C' := x_{i+1}Hx_iC[x_i, d_i]d_iP_Ry\overline{C}[y, x_i^{(+)}]x_i^{(+)}P_Rx_i^{(-)}\overline{C}[x_i^{(-)}, x_{i+1}^{(+)}]$$

$$x_{i+1}^{(+)}P_Rx_{i+1}^{(-)}\overline{C}[x_{i+1}^{(-)}, w]wP_RuC[u, x_{i+1}].$$

From Claim 2, the subgraphs $G[R(x_i^{(-)+}) \cup \cdots \cup R(x_i^-)]$, $G[R(d_i^+) \cup \cdots \cup R(x_i^{(+)-})]$, $G[R(x_{i+1}^{(-)+}) \cup \cdots \cup R(u^-)]$ and $G[R(x_{i+1}^+) \cup \cdots \cup R(x_{i+1}^{(+)-})]$ are pairwise remote and from Remark 2, they have less than λ vertices in each. On the other hand, according to the choice of y and w, the subgraph $G[R(y^+) \cup R(y^{++}) \cup \cdots \cup R(w^-)]$ and the previous subgraphs are pairwise remote. Then $|G[R(y^+) \cup R(y^{++}) \cup \cdots \cup R(w^-)]| \geqslant \lambda$, otherwise the cycle C' contradicts the choice of C. Consequently, $x_{i+1}^{(-)} \notin D_\lambda^+(y^+)$ and y^+ is as required.

If $u \in C[x_{i+1}^{(-)+}, x_{i+1}]$ and $d_i \in C[x_i^{(-)+}, x_i^-]$, let

$$C'' := x_{i+1}Hx_i\overline{C}[x_i, d_i]d_iP_Ry\overline{C}[y, x_i^{(+)}]x_i^{(+)}P_Rx_i^{(-)}\overline{C}[x_i^{(-)}, x_{i+1}^{(+)}]$$

$$x_{i+1}^{(+)}P_Rx_{i+1}^{(-)}\overline{C}[x_{i+1}^{(-)}, w]wP_RuC[u, x_{i+1}].$$

Using the same arguments as in the first case, we may prove the result. Notice that if $d_i = x_i^{(-)}$, we get a similar contradiction by replacing, in the cycle C'', the path $\overline{C}[x_i, d_i]d_iP_Ry\overline{C}[y, x_i^{(+)}]x_i^{(+)}P_Rx_i^{(-)}$ by $C[x_i, y]yP_Rx_i^{(-)}$.

In the case $u \in C[x_{i+1}^+, x_{i+1}^{(+)-}]$ and $d_i \in C[x_i, x_i^{(+)-}]$ and the case $u \in C[x_{i+1}^+, x_{i+1}^{(+)-}]$ and $d_i \in C[x_i^{(-)+}, x_i^-]$, we may get a similar proof as in the previous cases. Finally, y^+ is as required in the Claim. $\square$

From Claim 4, let b_i be the first vertex in $C[x_i^{(+)+}, x_{i+1}^{(-)-}]$ such that $x_{i+1}^{(-)} \notin D_\lambda^+(b_i)$ and for any $u \in D_\lambda^+(b_i) \cap C$ and any $v \in C[x_i^{(-)}, x_i^{(+)-}]$, there is no path uP_Rv. This implies that for any $u \in C[x_i^{(+)+}, b_i^-]$ there exists some $u' \in D_\lambda^+(u) \cap C$ which is connected to some $v' \in C[x_i^{(-)}, x_i^{(+)-}]$ by a path $u'P_Rv'$. In particular, there exists some $v \in C[x_i^{(-)}, x_i^{(+)-}]$ such that a path $b_i^-P_Rv$ exists.

Let $S_0 := H_\lambda(x_1)$ and $S_i := D_\lambda^+(b_i)$ for $1 \leqslant i \leqslant d$.

Claim 5. *For each $i \neq j$,*

(a) *For each $u \in C[x_i^{(-)+}, x_i^{(+)-}] - \{x_i\}$ and each $v \in C[x_j^{(-)+}, b_j^{--}]$, there is no path uP_Rv;*

(b) *$e(S_i, C[x_j^{(-)+}, b_j^-]) = 0$ if $i, j \in \{1, 2, \ldots, d\}$;*

(c) *S_i and S_j are remote if $i, j \in \{0, 1, \ldots, d\}$.*

Proof. To prove (a), suppose that $a_j \in C[x_j^{(-)+}, b_j^{--}]$ is the last vertex before b_j^- which is connected to some $w \in C[x_i^{(-)+}, x_i^{(+)-}] - \{x_i\}$ by a path a_iP_Rw. Necessarily, a_j belongs to $C[x_j^{(+)}, b_j^{--}]$ from Claim 2. Let $d_j \in C[a_j^+, b_j^-]$ be the first vertex after a_j which is connected to a vertex $v \in C[x_j^{(-)+}, x_j^{(+)-}]$ by a path d_jP_Rv. One of the four following cases must occur.

- If $v \in C[x_j, x_j^{(+)-}]$ and $w \in C[x_i^{(-)+}, x_i^-]$, let

$$C' := x_jHx_i\overline{C}[x_i, w]wP_Ra_j\overline{C}[a_j, x_j^{(+)}]x_j^{(+)}P_Rx_j^{(-)}\overline{C}[x_j^{(-)}, x_i^{(+)}]$$

$$x_i^{(+)}P_Rx_i^{(-)}\overline{C}[x_i^{(-)}, d_j]d_jP_Rv\overline{C}[v, x_j].$$

From Claim 2, and the choice of a_j and d_j the subgraphs $G[R(x_i^{(-)+}) \cup R(x_i^{(-)++}) \cup \cdots \cup R(w_i^-)]$, $G[R(x_i^+) \cup R(x_i^{++}) \cup \cdots \cup R(x_i^{(+)-})]$, $G[R(x_j^{(-)+}) \cup R(x_j^{(-)++}) \cup \cdots \cup R(x_j^-)]$, $G[R(v^+) \cup R(v^{++}) \cup \cdots \cup R(x_j^{(+)-})]$ and $G[R(a_j^+) \cup R(a_j^{++}) \cup \cdots \cup R(d_j^-)]$ are pairwise remote. Clearly from Remark 2, the subgraphs $G[R(x_i^{(-)+}) \cup \cdots \cup R(w_i^-)]$, $G[R(x_i^+) \cup \cdots \cup R(x_i^{(+)-})]$, $G[R(x_j^{(-)+}) \cup \cdots \cup R(x_j^-)]$, $G[R(v^+) \cup R(v^{++}) \cup \cdots \cup R(x_j^{(+)-})]$ have less than λ vertices in each. On the other hand, $|R(a_j^+) \cup \cdots \cup R(d_j^-)| < \lambda$ because otherwise, a_j^+ contradicts the choice of b_j which is the first vertex in $C[x_i^{(+)+}, x_{i+1}^{(-)-}]$ such that $x_{i+1}^{(-)} \notin D_\lambda^+(b_i)$ and for any $u \in D_\lambda^+(b_i)$ and any $v \in C[x_i^{(-)}, x_i^{(+)-}]$, there is no path uP_Rv. We conclude that the cycle C' contradicts the choice of C.

- If $v \in C[x_j, x_j^{(+)-}]$ and $w \in C[x_i^+, x_i^{(+)-}]$, let

$$C'' := x_jHx_iC[x_i, w]wP_Ra_j\overline{C}[a_j, x_j^{(+)}]x_j^{(+)}P_Rx_j^{(-)}\overline{C}[x_j^{(-)}, x_i^{(+)}]$$

$$x_i^{(+)}P_Rx_i^{(-)}\overline{C}[x_i^{(-)}, d_j]d_jP_Rv\overline{C}[v, x_j].$$

Using the same arguments, the cycle C'' contradicts the choice of C.

In the case $v \in C[x_j^{(-)+}, x_j^-]$ and $w \in C[x_i^{(-)+}, x_i^-]$ and the case $v \in C[x_j^{(-)+}, x_j^-]$ and $w \in C[x_i, x_i^{(+)-}]$ we may get a similar proof as in the previous cases.

To prove (b), suppose that a_i is the first vertex in $S_i \cap C$ which is connected to a vertex $w \in C[x_j^{(-)+}, b_j^-]$ by a path a_iP_Rw. If the vertex $v \in C[x_i^{(-)}, x_i^{(+)-}]$ which is connected to b_i^- by a path $vP_Rb_i^-$ belongs to $C[x_i, x_i^{(+)-}]$, one of the three main cases must occur.

• If $w \in C[x_j^{(-)+}, x_j^-]$, let

$$C_1 := x_jHx_iC[x_i, v]vP_Rb_i^-\overline{C}[b_i^-, x_i^{(+)}]x_i^{(+)}P_Rx_i^{(-)}\overline{C}[x_i^{(-)}, x_j^{(+)}]$$

$$x_j^{(+)}P_Rx_j^{(-)}\overline{C}[x_j^{(-)}, a_i]a_iP_RwC[w, x_j].$$

We know from Claim 2 that $G[R(x_i^{(-)+}) \cup R(x_i^{(-)++}) \cdots \cup R(x_i^-)]$, $G[R(v^+) \cup R(v^{++}) \cup \cdots \cup R(x_i^{(+)-})]$, $G[R(x_j^{(-)+}) \cup R(x_j^{(-)++}) \cup \cdots \cup R(w^-)]$ and $G[R(x_j^+) \cup R(x_j^{++}) \cup \cdots \cup R(x_j^{(+)-})]$ are pairwise remote. Since a_i is the first vertex in $S_i \cap C$ which is connected to a vertex $w \in C[x_j^{(-)+}, b_j^-]$, and from the definition of b_i, the subgraphs $G[R(b_i) \cup R(b_i^+) \cup \cdots \cup R(a_i^-)]$, $G[R(x_i^{(-)+}) \cup \cdots \cup R(x_i^-)]$, $G[R(v^+) \cup R(v^{++}) \cup \cdots \cup R(x_i^{(+)-})]$, $G[R(x_j^{(-)+}) \cup \cdots \cup R(w^-)]$ and $G[R(x_j^+) \cup \cdots \cup R(x_j^{(-)-})]$ are pairwise remote. Clearly, all these subgraphs have less than λ vertices in each, so the cycle C_1 contradicts the choice of C. Notice that we have assumed that $b_i \neq x_i^{(+)+}$, otherwise the contradiction is more easy to get, using $C[x_i, x_i^{(+)}]$ in the place of the path $C[x_i, v]vP_Rb_i^-\overline{C}[b_i^-, x_i^{(+)}]$ in the construction of C_1.

• If $w \in C[x_j, x_j^{(+)}]$, let

$$C_2 := x_jHx_iC[x_i, v]vP_Rb_i^-\overline{C}[b_i^-, x_i^{(+)}]x_i^{(+)}P_Rx_i^{(-)}\overline{C}[x_i^{(-)}, x_j^{(+)}]$$

$$x_j^{(+)}P_Rx_j^{(-)}\overline{C}[x_j^{(-)}, a_i]a_iP_Rw\overline{C}[w, x_j].$$

Using the same arguments as in the first case, the cycle C_2 contradicts the choice of C. Notice that if $w = x_j^{(+)}$, we get a similar contradiction by replacing, in the cycle C_2, the path $x_j^{(+)}P_Rx_j^{(-)}\overline{C}[x_j^{(-)}, a_i]a_iP_Rw\overline{C}[w, x_j]$ by $x_j^{(+)}P_Ra_iC[a_i, x_j]$.

• If $w \in C[x_j^{(+)+}, b_j^-]$, let u be the last vertex before w which is connected to a vertex $u' \in C[x_j^{(-)+}, x_j^{(+)-}]$ by a path uP_Ru'. We can see two subcases:

– If $u' \in C[x_j^{(-)+}, x_j]$, let

$$C_3 := x_jHx_iC[x_i, v]vP_Rb_i^-\overline{C}[b_i^-, x_i^{(+)}]x_i^{(+)}P_Rx_i^{(-)}\overline{C}[x_i^{(-)}, w]wP_Ra_i$$

$$C[a_i, x_j^{(-)}]x_j^{(-)}P_Rx_j^{(+)}C[x_j^{(+)}, u]uP_Ru'C[u', x_j].$$

We know that the subgraphs $G[R(x_i^{(-)+}) \cup \cdots \cup R(x_i^-)]$, $G[R(v^+) \cup R(v^{++}) \cup \cdots \cup R(x_i^{(+)-})]$, $G[R(x_j^{(-)+}) \cup \cdots \cup R(u'^-)]$, $G[R(x_j^+) \cup \cdots \cup R(x_j^{(+)-})]$, and $G[R(b_i) \cup R(b_i^+) \cup \cdots \cup R(a_i^-)]$ are pairwise remote and have less than λ vertices in each. From result (a) and the choice of u, we deduce that all these subgraphs and the subgraph $G[R(u^+) \cup R(u^{++}) \cup \cdots \cup R(w^-)]$ are pairwise remote. On the other hand, $|G[R(u^+) \cup R(u^{++}) \cup \cdots \cup R(w^-)]| < \lambda$, otherwise u^+ contradicts the choice of b_i. So the cycle C_3 contradicts the choice of C.

– If $u' \in C[x_j^+, x_j^{(+)-}]$, let

$$C_4 := x_j H x_i C[x_i, v] v P_R b_i^- \overline{C}[b_i^-, x_i^{(+)}] x_i^{(+)} P_R x_i^{(-)} \overline{C}[x_i^{(-)}, w] w P_R a_i$$

$$C[a_i, x_j^{(-)}] x_j^{(-)} P_R x_j^{(+)} C[x_j^{(+)}, u] u P_R u' \overline{C}[u', x_j].$$

Using the same arguments, the cycle C_4 contradicts the choice of C.

If the vertex $v \in C[x_i^{(-)}, x_i^{(+)-}]$ which is connected to b_i^- by a path $vP_Rb_i^-$ belongs to $C[x_i^{(-)}, x_i^-]$, we achieve the proof of assumption (b) by using the same arguments as in the previous case.

Now to prove (c) (which is evident if $i = 0$ or $j = 0$), assume that a_i is the first vertex in $S_i \cap C$ which is connected to S_j by a path P_R, and let a_j be the first vertex in $S_j \cap C$ such that there exists a path $a_iP_Ra_j$. Without loss of generality, suppose that the vertex $v_i \in C[x_i^{(-)+}, x_i^{(+)-}]$ which is connected to b_i^- by a path $v_iP_Rb_i^-$ is in $C[x_i^+, x_i^{(+)-}]$ and that the vertex $v_j \in C[x_j^{(-)+}, x_j^{(+)-}]$ which is connected to b_j^- by a path $v_jP_Rb_j^-$ is in $C[x_j^{(-)+}, x_j]$, then let

$$C' := x_j H x_i C[x_i, v_i] v_i P_R b_i^- \overline{C}[b_i^-, x_i^{(+)}] x_i^{(+)} P_R x_i^{(-)} \overline{C}[x_i^{(-)}, a_j] a_j P_R a_i$$

$$C[a_i, x_j^{(-)}] x_j^{(-)} P_R x_j^{(+)} C[x_j^{(+)}, b_j^-] b_j^- P_R v_j C[v_j, x_j].$$

From Claim 2 and result (a), $G[R(x_i^{(-)+}) \cup \cdots \cup R(x_i^-)]$, $G[R(v_i^+) \cup R(v_i^{++}) \cup \cdots \cup R(x_i^{(+)-})]$, $G[R(b_i) \cup R(b_i^+) \cup \cdots \cup R(a_i^-)]$, $G[R(x_j^{(-)+}) \cup \cdots \cup R(v_j^-)]$, $G[R(x_j^+) \cup \cdots \cup R(x_j^{(+)-})]$ and $G[R(b_j) \cup R(b_j^+) \cup \cdots \cup R(a_j^-)]$ are pairwise remote. From Remark 2 and the choice of a_i and a_j the previous subgraphs have less than λ vertices in each. So the cycle C' contradicts the choice of C. $\square$

Claim 6. *For each $0 < i < j$, $N(S_i)^- \cap N(S_j) \cap C[b_i, b_j] = \emptyset$.*

Proof. Suppose that the claim is not true and for some i, let a_i be a vertex in $S_i \cap C$ such that there exists some $a_j \in S_j \cap C$ with $N(\{a_i\})^- \cap N(\{a_j\}) \cap C[b_i, b_j] \neq \emptyset$, then let $d \in N(\{a_i\})^- \cap N(\{a_j\}) \cap C[b_i, b_j]$. Necessarily, d belongs to $C[b_i, x_j^{(-)-}]$ from Claim 5(b). Without loss of generality, we assume that the vertex $v_i \in C[x_i^{(-)+}, x_i^{(+)-}]$ which is connected to b_i^- by a path $v_iP_Rb_i^-$ is in $C[x_i, x_i^{(+)-}]$ and the vertex $v_j \in C[x_j^{(-)+}, x_j^{(+)-}]$ which is connected to b_j^- by a path $v_jP_Rb_j^-$ is in $C[x_j^{(-)+}, x_j]$. Let

$$C' := x_j H x_i C[x_i, v_i] v_i P_R b_i^- \overline{C}[b_i^-, x_i^{(+)}] x_i^{(+)} P_R x_i^{(-)} \overline{C}[x_i^{(-)}, a_j] a_j d \overline{C}[d, a_i] a_i d^+$$

$$C[d^+, x_j^{(-)}] x_j^{(-)} P_R x_j^{(+)} C[x_j^{(+)}, b_j^-] b_j^- P_R v_j C[v_j, x_j].$$

From Claim 2, the subgraphs $G[R(x_i^{(-)+}) \cup \cdots \cup R(x_i^-)]$, $G[R(v_i^+) \cup R(v_i^{++}) \cup \cdots \cup R(x_i^{(+)-})]$, $G[R(x_j^{(-)+}) \cup \cdots \cup R(v_j^-)]$ and $G[R(x_j^+) \cup \cdots \cup R(x_j^{(+)-})]$ are pairwise remote. According to the definition of S_i, we also know that $G[R(b_i) \cup R(b_i^+) \cup \cdots \cup R(a_i^-)]$, $G[R(x_i^{(-)+}) \cup \cdots \cup R(x_i^-)]$, and $G[R(v_i^+) \cup R(v_i^{++}) \cup \cdots \cup R(x_i^{(+)-})]$ are pairwise remote and the same holds for $G[R(b_j) \cup R(b_j^+) \cup \cdots \cup R(a_j^-)]$, $G[R(x_j^{(-)+}) \cup \cdots \cup R(v_j^-)]$ and $G[R(x_j^+) \cup R(x_j^{++}) \cup \cdots \cup R(x_j^{(+)-})]$. Finally, from Claim 5, all these subgraphs are pairwise remote and clearly they have less than λ vertices in each. So the cycle C' contradicts the choice of C. $\square$

Proof of Theorem 3 (*Conclusion*). First let us consider two subgraphs S_0 and S_1 and count the degree sum $d(S_0) + d(S_1)$.

We have $N_C(S_0) \subseteq \{x_1, x_2, \ldots, x_d\}$ and $N_R(S_0) - S_0 \subseteq H - S_0$. So $d(S_0) \leqslant d + |H| - \lambda$. On the other hand, by Claim 4, $x_1 \notin S_1$ and $N(S_1) \cap H = \emptyset$. By Claim 5(c), we have $N(S_1) \cap S_j = \emptyset$ for any $j \neq 1$. Furthermore, by the choice of b_1 and Claim 5(b) we have, respectively, $N(S_1) \cap C[x_1^{(-)}, x_1^{(+)-}] = \emptyset$ and $N(S_1) \cap C[x_j^{(-)+}, b_j^-] = \emptyset$ for any $j \neq 1$. Then

$$d(S_1) = |N(S_1) - S_1| \leqslant n - |H| - \sum_{j=1}^{d} |S_j| - |C[x_1^{(-)}, x_1^{(+)-}]| - \sum_{j=2}^{d} |C[x_j^{(-)+}, b_j^-]|.$$

We know that $|C[x_1^{(-)}, x_1^{(+)-}]| \geqslant 2, |C[x_j^{(-)+}, b_j^-]| \geqslant 2$ and $|S_j| = \lambda$ for any j, so

$$d(S_1) \leqslant n - |H| - d\lambda - 2 - 2(d-1).$$

Then,

$$d(S_0) + d(S_1) \leqslant n - (d+1)\lambda - d \leqslant n - (k+1)\lambda - k,$$

which contradicts the hypothesis of Theorem 3.

Now let us consider three subgraphs S_0, S_1, S_2 and count the degree sum $\sum_{i=0}^{2} d(S_i)$.

By Claim 5(c), $N_R(S_1) \cap N_R(S_2) = \emptyset$. For every $1 \leqslant i \leqslant d$, let $y_i \in C$ such that y_i^- is the last vertex of $S_i \cap C$ and define $L_i = C[y_i, x_{i+1}^{(-)}]$, $L = \bigcup_{i=1}^{d} L_i$ and $M = V(G) - L$.

From Claims 4 and 5, we have $N(S_1) \cap H = \emptyset$, $N(S_1) \cap S_i = \emptyset$ for any $i \neq 1$ and $(N_C(S_1) - S_1) \subseteq (\bigcup_{i=1}^{d} L_i) \cup C[x_1^{(+)}, b_1^-]$. By the same reason we also have $N(S_2) \cap H = \emptyset$, $N(S_1) \cap S_i = \emptyset$ for any $i \neq 2$ and $(N_C(S_2) - S_2) \subseteq (\bigcup_{i=1}^{d} L_i) \cup C[x_2^{(+)}, b_2^-]$.

It follows that

$$\begin{aligned} d_M(S_1) + d_M(S_2) &\leqslant n - |\bigcup_{i=1}^{d} L_i| - \sum_{i=1}^{d} |S_i| - |H| - \sum_{i=3}^{d} |C[x_i^{(-)+}, b_i^-]| \\ &\quad - |C[x_1^{(-)+}, x_1^{(+)-}]| - |C[x_2^{(-)+}, x_2^{(+)-}]| \\ &\leqslant n - |\bigcup_{i=1}^{d} L_i| - |H| - d\lambda - 2(d-2) - 2. \end{aligned}$$

On the other hand, it results from Claim 6 that $|N_{L_i}(S_1)| + |N_{L_i}(S_2)| \leqslant |L_i| + 1$ for $2 \leqslant i \leqslant d$. Moreover by Claim 4, $x_2^{(-)} \notin N(S_2)$, so we have $|N_{L_1}(S_1)| + |N_{L_1}(S_2)| \leqslant |L_1|$ We now sum over i and obtain

$$d_L(S_1) + d_L(S_2) = \sum_{i=1}^{d} (|N_{L_i}(S_1)| + |N_{L_i}(S_2)|) \leqslant |\bigcup_{i=1}^{d} L_i| + d - 1.$$

Therefore

$$d(S_1) + d(S_2) \leqslant n - |H| - d\lambda - d + 1.$$

On the other hand, as in the case $t = 2$, we have

$$d(S_0) \leqslant d + |H| - \lambda.$$

So

$$\sum_{i=0}^{2} d(S_i) \leqslant n - (d+1)\lambda + 1 \leqslant n - (k+1)\lambda + 1,$$

which contradicts the hypothesis and achieves the proof of Theorem 3. □

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